Optimal Punishments in Linear Duopoly Supergames with Product Di®erentiation[1]

Luca Lambertini[x] - Dan Sasaki[#]
x Department of Economics
University of Bologna
Strada Maggiore 45
40125 Bologna, Italy
fax 0039-51-6402664
e-mail lamberti@spbo.unibo.it
Economics Department
University of Melbourne
Parkville, Victoria 3052, Australia
fax 0061-3-93446899
e-mail dsasaki@cupid.ecom.unimelb.edu.au

September 18, 1998

[1]We thank Christian Schultz, the audience at the XXV EARIE Conference (Copenhagen, August 27-30, 1998), and three anonymous referees for useful comments. The usual disclaimer applies.

Abstract

We analyse optimal penal codes in both Bertrand and Cournot supergames with product di®erentiation. We prove that the relationship between optimal punishments and the security level (individually rational discounted pro¯t stream) depends critically on the degree of supermodularity in the stage game, using a linear duopoly supergame with product di®erentiation. The security level in the punishment phase is reached only under extreme supermodularity, i.e., when products are perfect substitutes and ¯rms are price setters. Finally, we show that Abreu's rule cannot be implemented under Cournot behaviour and strong demand complementarity between products.

Keywords: penal codes, security level, product di®erentiation, positivity constraints

JEL classi¯cation: C72, D43, L13

1 Introduction

Optimal punishment, originated by Abreu (1986; 1988), has collected broad attention, yet generated surprisingly scanty applications in the industrial organisation literature despite the large number of contributions in the theory of collusion. One of the few attempts in this direction is Lambson (1987), investigating the relationship between the optimal penal codes and the "security level", i.e. the discounted flow of profits at which the participation constraint just binds. Using a Bertrand supergame with homogeneous products, Lambson finds that the optimal punishment drives firms indifferent between continuing and discontinuing the supergame. In more recent contributions (Lambson, 1994; 1995) it is shown that, if firms' *a priori* symmetry is waived, optimal punishments may no longer hit the security level. Specifically, if firms differ in size, the security level punishment is operative for the large firms but not for the smaller ones.

On the other hand, there exists a wide literature concerning the effects of the amount of product differentiation, on the stability of implicit collusion either in output levels or in prices (Deneckere, 1983; Chang, 1991, 1992; Rothschild, 1992; Ross, 1992; Friedman and Thisse, 1993; Häckner, 1994, 1995, 1996; Lambertini, 1997, inter alia). All this literature, apart from Häckner (1996), uses the traditional Friedman (1971) formulation of folk theorem.

Our effort in this paper is to investigate the bearings of product differentiation on optimal punishments, both in quantity- and price-setting supergames. The optimal punishment, as well as the associated critical threshold in the discount factor, depends on the slope of the firms' best reply functions in the stage game. In particular, we discover that Lambson's finding is sensitive to the degree of substitutability between firms' products. The security level, i.e. the discounted participation constraint, binds exclusively when products are perfect substitutes under price competition. The intuition behind this result is that extreme supermodularity forces firms to accept their lowest individually rational payoff independently of the duration of the game. A one-shot-game analogue of this result is the well known Bertrand-Nash equilibrium with perfect substitute products where firms would obtain zero profits, thereby as far as their participation to the game is concerned, they would be kept at the indifference condition. Moreover, we find that, if products are fairly close complements in demand, collusion can be sustained by optimal punishments only under price behaviour. Under Cournot behaviour, the output expansion required to inflict the punishment produces the effect of increasing the incentive to deviate from the penal code to such an extent that the punishment itself is not implementable.

The paper is organised as follows. The basic model is laid out in section 2. Then, Bertrand and Cournot supergames are analysed in sections 3 and 4, respectively, as they lead to qualitatively distinct results. Section 5 provides a brief discussion on game-theoretic similarities and differences between price competition and quantity competition. Finally, section 6 concludes the paper.

2 The setup

Two firms operate on the market, selling possibly differentiated products. Each firm faces the following inverse demand function:

$$p_i = 1 - q_i - \gamma q_j \qquad \{i, j\} = \{1, 2\};\ q_1 \geq 0;\ q_2 \geq 0 \tag{1}$$

in which $\gamma \in (-1, 1]$ measures the degree of differentiation (see Dixit, 1979; Singh and Vives, 1984).

The marginal production cost is constant and thus normalised to zero. Or, more precisely, what we refer to as the "price" in this paper is in fact the price minus the marginal production cost. We allow the possibility that this "price" may fall below zero, similarly to Lambson (1987).[1] On the other hand, production quantities should stay non-negative for obvious technological constraints.

We consider either a Bertrand supergame or a Cournot supergame, where this demand structure appears unchanged in every stage game. The discount factor $\delta \in [0, 1)$ is common to both firms. In a Cournot supergame, each firm chooses a positive quantity $q_i \geq 0$ at the beginning of every stage, and then the market prices realise according to (1).

In a Bertrand supergame, at the beginning of every stage game, the two firms simultaneously set their prices p_1, p_2. By inverting (1), the direct demand function obtains:

$$q_i = \frac{1}{1+\gamma} - \frac{1}{1-\gamma^2}p_i + \frac{\gamma}{1-\gamma^2}p_j \tag{2}$$

as long as $\gamma < 1$. If these prices result in positive quantities $\tilde{q}_1 \geq 0$, $\tilde{q}_2 \geq 0$ according to (2), then these quantities realise unmodified. Otherwise, if $\tilde{q}_i = \min\{\tilde{q}_1, \tilde{q}_2\} < 0$, the non-negative quantity constraint forces the actual quantity realisation to be $q_i = 0$ and then q_j is determined by (1) setting $q_i = 0$, or, if this q_j is also negative, then realised quantities are $q_1 = q_2 = 0$.

Note that, as $\gamma \uparrow 1$, the demand function (2) approaches the undifferentiated Bertrand demand function. Hereby without infringing the continuity of the model, we can assume that when $\gamma = 1$ and $p_i \leq p_j$ the whole demand would be taken by firm i whilst firm j sells nil, and that when $\gamma = 1$ and $p_i = p_j$ the two firms share the demand evenly.

Throughout this paper, our main focus is to inspect Abreu's (1986; 1988) optimal punishment rule. Firms initially follow a prescribed collusive path, until any deviation is detected. We assume that firms agree to collude at the Pareto

[1] If the marginal production cost is relatively low, and if we stipulate that consumer prices be nonnegative, then such a nonnegativity constraint may still remain relevant to our analysis. In this paper we treat the "price" (in excess of the marginal production cost) to be effectively unbounded, much in the same vein as Lambson (1987).

frontier of joint profit maximisation, and to split the profits symmetrically. Therefore, the collusive price is the monopoly price p^M, and each firm's collusive profit π^M and quantity q^M are half the monopoly profit and quantity, respectively:

$$p^M = \frac{1}{2}; \quad q^M = \frac{1}{2(1+\gamma)}; \quad \pi^M = \frac{1}{4(1+\gamma)}. \tag{3}$$

Abreu (1986) proves that, if there exist an action a^P (either price p^P or quantity q^P) and a discount factor $\delta^*_K(\gamma)$ satisfying the following system of equations, then there exists no punishment rule which can sustain collusion at (3) when $\delta < \delta^*_K(\gamma)$:

$$\pi^D_K(a^M) - \pi^M = \delta^*_K(\gamma)(\pi^M - \pi_K(a^P)) \tag{4}$$
$$\pi^D_K(a^P) - \pi_K(a^P) = \delta^*_K(\gamma)(\pi^M - \pi_K(a^P)) \tag{5}$$

where $\pi_K(a^P)$ denotes each firm's stage profit when both firms play a^P, whilst $\pi^D_K(a^M)$ is the profit from a one-shot best deviation from the collusive path, and finally $\pi^D_K(a^P)$ is the profit from a one-shot best response against a^P. We use capitalised subscripts to indicate the form of market competition: $K = B$ for price competition (Bertrand) or $K = C$ for quantity competition (Cournot).

Whenever $\delta \geq \delta^*_K(\gamma)$, collusion at (3) is sustainable through the following penal code. If a deviation is detected in period t, then in the next period $t+1$, firms switch to a punishment phase where both firms adopt the punishment action a^P irrespective of which firm is punishing the other. If both firms follow the prescribed penal code at $t+1$, then they revert to the initial collusive path from $t+2$ onwards. Otherwise, the punishment phase continues until the penal code is adopted by both firms at the same time. Abreu (1986) discovers that this symmetric penal code, when a^P satisfies the system of equations (4-5), is optimal in that it requires a lower discount factor to sustain the collusion than any other punishment rule. Following Abreu (1986) and Häckner (1996), throughout this paper we concentrate on this symmetric punishment in that both firms take the same action in the penal period.

Observe that the incentives not to deviate (i) from the collusive path initially agreed upon, and (ii) from the optimal symmetric penal code, are described by:

$$\pi^D_K(a^M) - \pi^M \leq \delta(\pi^M - \pi_K(a^P)); \tag{6}$$
$$\pi^D_K(a^P) - \pi_K(a^P) \leq \delta(\pi^M - \pi_K(a^P)). \tag{7}$$

The aforementioned one-shot punishment rule satisfies both of these conditions with strict equalities.

Although the system (4-5) consists of two equations with two unknowns, the solution $\left\{a^P; \delta^*_K(\gamma)\right\}$ may not always exist since quantities should always be non-negative. Namely,

- Either in a Cournot supergame when the non-negativity of quantities is unbinding, or in a Bertrand supergame, the solution always exists. Hence, Abreu's rule is operative.

- In a Cournot supergame, once the non-negativity of quantities binds, then the system (4-5) has no positive-quantity solution, which renders Abreu's rule technologically unimplementable. Therefore in order to sustain collusion under this circumstance, either multi-period penal code or partial collusion (in the sense that firms collude on a profile less profitable than the monopoly level) ought to be sought.

On the other hand, the penal code a^P is allowed to be severe enough to drive prices and therefore profits below zero in the punishment phase. However, the individual participation constraint

$$\pi_K(a^P) + \sum_{\tau=1}^{\infty} [\delta_K^*(\gamma)]^{\tau} \pi^M \geq 0 \tag{8}$$

must be satisfied for all admissible values of γ, for firms to be willing to continue the supergame after any deviation from the initial collusive path. For, if constraint (8) were violated, firms would find it preferable to abandon production permanently, because the intensity of the punishment would overbalance the discounted value of collusive profits. In particular, in a Bertrand supergame with $\gamma = 1$, Lambson (1987) shows that the optimal punishment drives each firm down to the "security level". Namely, each firm's discounted profit stream commencing in a punishment period (the left-hand side of inequality 8) equals zero, which is indeed each firm's individually rational profit at which the firm is just indifferent between obeying the prescribed penal code and shutting down its production activity forever. We shall discover in the following two sections that this observation is extremely fragile in other kinds of supergames.

For future reference, it is useful to investigate the degree of strategic complementairty/substitutability characterising the constituent stage games in the two alternative settings. Obviously, strategic complementarity/substitutability is strictly related to supermodularity, i.e., the second derivative of payoff functions (see Bulow, Geanakoplos and Klemperer, 1985; and, for a wide overview on supermodular games, chapter 12 in Fudenberg and Tirole, 1991). Namely, the slope of firm i's reaction function is $\partial^2\pi_K/\partial a_i \partial a_j$; unless positivity constraints bind. It is easily verified that

$$\frac{\partial^2 \pi_C}{\partial q_i \partial q_j} = -\gamma ; \tag{9}$$

i.e., the second cross derivative of a one-shot Cournot profit function is a negatively 45°-sloped line. In the case of Bertrand behaviour, we have

$$\frac{\partial^2 \pi_B}{\partial p_i \partial p_j} = \frac{\gamma}{1-\gamma^2} ; \tag{10}$$

which is, on the contrary, an increasing function of γ. It is then promptly verified that the maximum degree of supermodularity (which we label 'perfect' or 'complete' supermodularity in the remainder of the paper) is reached either at $\gamma = -1$ under Cournot behaviour, or at $\gamma = 1$ under Bertrand behaviour.

3 Differentiated Bertrand Supergame

The Bertrand supergame with optimal penal codes can be characterised as follows.

Proposition I:

- Optimal symmetric punishment leads to the security level payoff if and only if $\gamma = 1$. Otherwise, if $\gamma \in (-1;1)$, the discounted profits from the punishment period onwards is strictly above the security level.
- The non-negativity constraint on the quantity being supplied by the cheated firm during the deviation period binds for $\gamma \in \left[\sqrt{3}-1;1\right]$.
- The optimal deviation in the punishment phase implies a zero profit for $\gamma \in \left[\frac{3\sqrt{5}-5}{2};1\right]$.

Proof: Under Bertrand competition, Abreu's rule becomes:

$$\pi_B^D(p^M) - \pi^M = \delta_B^*(\gamma)(\pi^M - \pi_B(p^P)); \tag{11}$$
$$\pi_B^D(p^P) - \pi_B(p^P) = \delta_B^*(\gamma)(\pi^M - \pi_B(p^P)): \tag{12}$$

The optimal punishment price p^P as well as the critical discount factor $\delta_B^*(\gamma)$ for collusive sustainability are determined as follows. See Appendix A.1 for computational details.

- Over the range $\gamma \in (-1;\sqrt{3}-1)$, the system (11-12) is solved by:

$$p^P = \frac{2-3\gamma}{2(2-\gamma)}; \quad \delta_B^* = \frac{(2-\gamma)^2}{16(1-\gamma)}: \tag{13}$$

- In the regime $\gamma \in \left[\sqrt{3}-1;\frac{3\sqrt{5}-5}{2}\right)$, the system (11-12) yields:

$$p^P = \frac{(1-\gamma)\gamma - \sqrt{-1+2\gamma-\gamma^3}}{(2-\gamma)\gamma}; \quad \delta_B^* = \frac{(2-\gamma)^2(\gamma^2+\gamma-1)}{(\gamma^2+2\sqrt{-1+2\gamma-\gamma^3})^2}: \tag{14}$$

- Over the region $\gamma \in \left[\frac{3\sqrt{5}-5}{2};1\right]$, the solution to (11-12) is:

$$p^P = \frac{1}{2} - \frac{\sqrt{2\gamma^2+\gamma-1}}{2\gamma}; \quad \delta_B^* = \frac{\gamma^2+\gamma-1}{2\gamma^2+\gamma-1}: \tag{15}$$

Finally, in order to verify the statement about the security level, it suffices to calculate the value of discounted profits from the punishment period onwards, in the three relevant ranges of γ as in (13) through (15). Namely,

$$\pi_B(p^P) + \sum_{\tau=1}^{\infty} [\delta_B^*(\gamma)]^\tau \pi^M > 0 \qquad \forall \gamma \in (-1;1); \tag{16}$$

i.e., the discounted flow of profits associated with the punishment is strictly positive over the generic range $\gamma \in (-1;1)$, so that the individual participation constraint to the continuation of the supergame does not bind.

When $\gamma = 1$, since firms are providing homogeneous products, $\pi_B^D = 2\pi^M$, $\delta_B^* = \frac{1}{2}$, and $p^P = \frac{1-\sqrt{2}}{2}$. Only in this situation, the punishment leads to security level payoffs (see Abreu, 1986; Lambson, 1987), i.e.,

$$\pi_B\left(\frac{1-\sqrt{2}}{2}\right) + \sum_{\tau=1}^{\infty} [\delta_B^*(1)]^\tau \pi^M = 0. \tag{17}$$

This concludes the proof. ∎

An intuitive interpretation :

The left-hand sides of (16) and (17) are the discounted profit streams starting from the punishment period. Therefore, the security level can be viewed as a dynamic participation constraint to the supergame.

This resembles what happens in a one-shot price game with homogeneous products, i.e., that extreme strategic complementarity drives the equilibrium price down to marginal cost and thereby the equilibrium profits to zero, which is indeed the participation constraint to the one-shot game.

Note finally that, when $\gamma \in \left(\frac{2}{3}; 1\right]$, the optimal punishment price p^P falls strictly negative. This marks a key difference between Bertrand and Cournot cases, as shall be discussed in the following two sections.

4 Differentiated Cournot Supergame

The Cournot supergame with optimal punishment leads to the following.

Proposition II : In sustaining collusion at q^M, Abreu's optimal punishment is operative when $\gamma \in \left(-\frac{2}{3}; 1\right]$. Over this range,

- The deviation from the collusive path never drives the opponent's quantity down to zero.
- The optimal punishment drives firms' quantities and thus one-shot profits down to zero when $\gamma = -\frac{2}{3}$, but never drives their discounted streams of profits down to the security level.

- the deviation from the optimal punishment always produces a strictly positive one-shot profit.

For any $\gamma \in \left(-1, -\frac{2}{3}\right)$, the system (4-5) has no solution and therefore Abreu's rule is inoperative in sustaining collusion at q^M.

Proof: Given that the stage game is Cournot, the relevant Abreu's rule is the following system of simultaneous equations:

$$\pi_C^D(q^M) - \pi^M = \delta_C^*(\gamma)(\pi^M - \pi_C(q^P)); \tag{18}$$

$$\pi_C^D(q^P) - \pi_C(q^P) = \delta_C^*(\gamma)(\pi^M - \pi_C(q^P)). \tag{19}$$

The solutions to this system are similar to the Bertrand case, except that those constraints concerning the positivity of output quantities bind over different ranges of γ. See Appendix A.2 for further details.

- When $\gamma \in \left[-\frac{2}{3}, 1\right)$, equations (18) and (19) are solved by

$$q^P = \frac{2+3\gamma}{2(1+\gamma)(2+\gamma)}; \quad \delta_C^* = \frac{(2+\gamma)^2}{16(1+\gamma)}. \tag{20}$$

- When $\gamma \in \left(-1, -\frac{2}{3}\right)$, the quantity q^P in (20) becomes negative, therefore the system of equations (18-19) has no positive-quantity solution. This indicates that Abreu's rule is unimplementable over this range of γ.

In order to verify the statement about the security level, it suffices to compute the value of discounted profits from the punishment period onwards over the range $\gamma \in \left[-\frac{2}{3}, 1\right)$, according to (20). Namely,

$$\pi_C(q^P) + \sum_{\tau=1} [\delta_C^*(\gamma)]^\tau \pi^M > 0 \qquad \forall \gamma \in \left[-\frac{2}{3}, 1\right), \tag{21}$$

i.e., the discounted flow of profits associated with the punishment is strictly positive regardless of the value of γ, so that the individual participation constraint to the continuation of the supergame never binds. ■

An intuitive interpretation:

In Cournot games, that range of γ where extreme strategic complementarity would draw the optimal penal code near the security level punishment should be in the neighbourhood of $\gamma \simeq -1$. However, when firms' products are mutually strongly complementary, the positivity constraint of output quantities prohibits firms from exercising severe enough punishment. This not only bounds the punishment payoffs away from the security level, but also affects the sustainability of collusion in the first place.

The intuition why collusive stability is unattainable by means of one-shot punishment under extreme complementarity is as follows. When $\gamma = -.999$, for instance, the collusive quantity is $q^M = 500$ and the collusive profit is $\pi^M = 250$. A one-shot deviation from this path can earn as much as 15718.890625, by retracting the quantity to $R_C(q^M) = 125.375$. On one hand, due to the non-negativity of quantities, any punishment involving contraction in production quantities cannot force the profit below zero, and thereby violates inequality (6) as well as equation (18) uniformly for any $\delta \in [0, 1)$. On the other hand, even though it is not impossible to achieve negative prices and profits by overproduction, due to extreme complementarity, such an overexpansionary penal path would only serve to increase the profitability of deviation from the penal path itself, which would violate inequality (7) and equation (19) instead. Hence, neither zero production nor overproduction can serve as a sustainable prescription for punishment.

5 Discussion on Price-Quantity Duality

In this section, we investigate the technical reason why price competition and quantity competition lead to sharply different results concerning optimal punishments, as has been shown in previous two sections. In general, the linear demand functions

$$p_i = 1 - q_i - \gamma q_j \qquad \{i, j\} = \{1, 2\}; \tag{22}$$

$$q_i = \frac{1}{1+\gamma} - \frac{1}{1-\gamma^2} p_i + \frac{\gamma}{1-\gamma^2} p_j \tag{23}$$

(see Singh and Vives, 1984) are convenient in maintaining the duality between price-setting and quantity-setting games by means of reparametrisation in γ, essentially by flipping its sign. This duality has been well known in the literature, and therefore connoisseurs may have been puzzled why in our paper, on the contrary, the two forms of competition entail substantially distinct game-theoretic characteristics.

Essentially, the asymmetry between price competition and quantity competition stems not from Abreu's optimal punishment rule, but purely from non-negativity constraints. For instance, Deneckere (1984) shows the similarity between Bertrand games with positive quantity constraints but without positive price constraints, and Cournot games with positive price constraints but without positive quantity constraints. He shows that the positivity of the cheated firm's quantity in the former game binds when $\gamma \geq \sqrt{3} - 1$, whereas the positivity of the cheated firm's price in the latter game binds when $\gamma \leq 1 - \sqrt{3}$. Note that, since

Deneckere's analogy between Cournot and Bertrand games is about the one-shot deviation from the collusive path, all the results there must be independent of what penal codes are employed in order to sustain the collusion. Instead, the key is: when we interchange firms' strategic variables between prices and quantities, we also need to interchange likewise all the relevant constraints such as non-negativity, in order to maintain the duality.

The interchangeability between positive quantity constraints and positive price constraints is, however, not always realistic when economics is concerned. Positivity constraints on prices can be loosened once we reconceptualise "prices" as the actual prices in excess of the marginal production costs, precisely as we have done in this paper. On the other hand, it is less straightforward how, if possible at all, to make similar reconceptualisation in quantities. This is the reason why we have arrived in a substantial asymmetry between our Bertrand (section 3) and Cournot (section 4) supergames.

If we strictly followed Deneckere by imposing positivity constraints only on prices but not on quantities in Cournot supergames, then unlike in section 3, Abreu's rule would prevail over the entire parametric range $\gamma \in (-1, 1]$. Analogous to the Bertrand case, the Cournot system of equations (18-19) would be solved by:

$$q^P = \frac{\gamma + \sqrt{2\gamma^2 - \gamma - 1}}{2(1+\gamma)\gamma}; \quad \delta^*_C = \frac{\gamma^2 - \gamma - 1}{2\gamma^2 - \gamma - 1} \quad \forall \gamma \in \left(-1; \frac{5 - 3\sqrt{5}}{2}\right]; \tag{24}$$

$$\left.\begin{aligned} q^P &= \frac{(1+\gamma)\gamma - \sqrt{\gamma^3 - 2\gamma - 1}}{(1+\gamma)(2+\gamma)\gamma} \\ \delta^*_C &= \frac{(\gamma+2)^2(\gamma^2 - \gamma - 1)}{(\gamma^2 + 2\sqrt{\gamma^3 - 2\gamma - 1})^2} \end{aligned}\right\} \quad \forall \gamma \in \left(\frac{5 - 3\sqrt{5}}{2}; 1 - \sqrt{3}\right]; \tag{25}$$

$$q^P = \frac{2 + 3\gamma}{2(1+\gamma)(2+\gamma)}; \quad \delta^*_C = \frac{(2+\gamma)^2}{16(1+\gamma)} \quad \forall \gamma \in \left(1 - \sqrt{3}; 1\right] \tag{26}$$

As $\gamma \downarrow -1$, both the collusive profit π^M and the one-shot deviation profit $\pi^D_C(q^M)$ diverge to infinity. In section 4, their rates of divergence were $(1+\gamma)^{-1}$ and $(1+\gamma)^{-2}$ respectively, the latter being disproportionately faster than the former. This was the intuition why one-shot punishment went out of operation. Now, due to the alteration in positivity constraints, the collusive profit π^M and the deviation profit $\pi^D_C(q^M)$ diverge to $+\infty$ as well as the punishment profit $\pi_C(q^P)$ to $-\infty$ all at the same order of divergence $(1+\gamma)^{-1}$. Therefore, rescaling them appropriately by

$$\Pi^M = (1+\gamma)\pi^M; \quad \Pi_C = (1+\gamma)\pi_C(q^P); \tag{27}$$

we would retrieve Lambson's (1987) result asymptotically, i.e.,

$$\lim_{\gamma \to -1} \left(\Pi_C + \sum_{\tau=1}^{\infty} [\delta_C^*(\gamma)]^{\tau} \Pi^M \right) = 0. \tag{28}$$

The security level would thereby be reached in Abreu's optimal punishment under extreme strategic complementarity, whether in prices or in quantities.

6 Concluding remarks

We have derived optimal penal codes and the associated critical discount factors in a duopoly supergame with differentiated products, under both Cournot and Bertrand behaviour. We have shown that the relationship between optimal punishments and the security level hinges critically upon the degree of supermodularity in the stage game which, in a duopoly supergame, is determined by the degree of product differentiation. The security level in the punishment phase is reached only under extreme supermodularity, i.e., when products are perfect substitutes and the basic stage game exhibits increasing best replies. Under extreme complementarity in demand, collusion in quantities at the monopoly output becomes unsustainable due to the impossibility of preventing deviation from the optimal symmetric penal code.

Appendix

A.1 Optimal Punishments in Bertrand Supergames

Define a firm's best reply against any price p set by the rival as $R_B(p)$. The algebraic form of this reaction function shifts across the following three regimes depending upon the positivity of firms' output quantities: 1. both firms sell positive quantities, 2. only the cheated (opponent) firm sells a positive quantity, 3. only the cheating firm sells a positive quantity. Theoretically there can be one more regime: 4. both firms sell nil, but this one is obviously irrelevant to our objective.

1. When quantities are positive, from the profit function

$$\pi_i = p_i q_i = p_i \left(\frac{1}{1+\gamma} - \frac{1}{1-\gamma^2} p_i + \frac{\gamma}{1-\gamma^2} p_j \right) \tag{29}$$

$$\frac{\partial \pi_i}{\partial p_i} = \frac{1}{1+\gamma} - \frac{2}{1-\gamma^2} p_i + \frac{\gamma}{1-\gamma^2} p_j ; \tag{30}$$

the best reply function and the resulting one-shot deviation profit obtain:

$$R_B(p) = \frac{1-\gamma+\gamma p}{2}; \qquad \pi_B^D(p) = \frac{(1-\gamma+\gamma p)^2}{4(1-\gamma^2)}. \tag{31}$$

with the resulting sales quantity $\dfrac{1-\gamma+\gamma p}{2(1-\gamma^2)}$.

2. When this form of price $R_B(p)$ in (31) becomes negative, which happens when and only when the resulting quantity would also become negative, the positivity constraint on quantities becomes binding, so that the maximum one-shot profit is simply

$$\pi_B^D(p) = 0. \tag{32}$$

3. On the other hand, when it is one-shot profit maximal to drive the opponent firm's output quantity down to nil, i.e.,

$$q_j = \frac{1}{1+\gamma} - \frac{1}{1-\gamma^2} p_j + \frac{\gamma}{1-\gamma^2} p_i = 0 \iff p_i = \frac{p_j - 1}{\gamma} - 1; \tag{33}$$

the required reaction against the opponent's price p and the resulting one-shot profit become

$$R_B(p) = 1 - \frac{1-p}{\gamma}; \quad \pi_B^D(p) = \frac{(1-p)(\gamma+p-1)}{\gamma^2}. \tag{34}$$

Clearly, the one-shot deviation from the initial collusive path cannot belong to regime (32). The border between the other two regimes (31) and (34) is $\gamma = \sqrt{3} - 1$, i.e., when $p = p^M = \frac{1}{2}$, the prices and quantities (31) and (34) mutually coincide.

As to the deviation from the punishment phase, regime (34) is clearly absent. The border between the other two, (31) and (32), depends upon the punishment price p^P, thus we must exhaust both possibilities henceforth.

In all of the following cases, the one-shot profit in the penal phase is

$$\pi_B(p^P) = \frac{(1 - p^P)p^P}{1 + \gamma}. \tag{35}$$

- When $\gamma \in (-1, \sqrt{3} - 1)$, assume first that the optimal deviation from the punishment phase also falls in regime (31). Thus, using

$$\pi_B^D(p^M) = \frac{(1 - \gamma + \gamma p^M)^2}{4(1 - \gamma^2)} = \frac{(2 - \gamma)^2}{16(1 - \gamma^2)}; \qquad (31)|p^M$$

$$\pi_B^D(p^P) = \frac{(1 - \gamma + \gamma p^P)^2}{4(1 - \gamma^2)}. \qquad (31)|p^P$$

  we solve the system of equations (11) and (12). This leads to

$$p^P = \frac{2 - 3\gamma}{2(2 - \gamma)}; \quad \delta_B^* = \frac{(2 - \gamma)^2}{16(1 - \gamma)}. \tag{13}$$

  Note that the one-shot best reaction against this penal code p^P, in fact, has the form (31) over the entire range $\gamma \in (-1, \sqrt{3} - 1)$.

- Now, proceed to the remainder of the substitutability range $\gamma \in [\sqrt{3} - 1, 1]$. Again, we assume first that the optimal one-shot reaction against p^P falls in regime (31) and solve simultaneous equations (11) and (12), based upon

$$\pi_B^D(p^M) = \frac{(1 - p^M)(\gamma + p^M - 1)}{\gamma^2} = \frac{2\gamma - 1}{4\gamma^2}; \qquad (34)|p^M$$

$$\pi_B^D(p^P) = \frac{(1 - \gamma + \gamma p^P)^2}{4(1 - \gamma^2)}. \qquad (31)|p^P$$

  This time, the solution

$$p^P = \frac{(1 - \gamma)\gamma + \sqrt{-1 + 2\gamma - \gamma^3}}{(2 - \gamma)\gamma}; \quad \delta_B^* = \frac{(2 - \gamma)^2(\gamma^2 + \gamma - 1)}{(\gamma^2 - 2\sqrt{-1 + 2\gamma - \gamma^3})^2} \tag{14}$$

  indeed falls in regime $(31)|p^P$ if and only if $\gamma < \dfrac{3\sqrt{5} - 5}{2}$.

- Finally, when $\gamma \in \left[\frac{3\sqrt{5}-5}{2}; 1\right]$, the optimal punishment and the critical value of the discount factor can be obtained by setting equal to zero the quantity of the firm being cheated, as well as the one-shot deviation profit from the punishment path. Namely,

$$\pi_B^D(p^M) = \frac{(1-p^M)(\gamma+p^M-1)}{\gamma^2} = \frac{2\gamma-1}{4\gamma^2}; \qquad (34)|_{p^M}$$

$$\pi_B^D(p^P) = 0. \qquad (32)|_{p^P}$$

The system of equations (11) and (12) is solved by:

$$p^P = \frac{1}{2} - \frac{\sqrt{2\gamma^2+\gamma-1}}{2\gamma}; \quad \delta_B^* = \frac{\gamma^2+\gamma-1}{2\gamma^2+\gamma-1}. \qquad (15)$$

A.2 Optimal Punishments in Cournot Supergames

Define a firm's one-shot best reply against any quantity q set by the rival as $R_C(q)$. Directly from the profit function

$$\pi_i = p_i q_i = (1 - q_i - \gamma q_j) q_i \qquad (36)$$

$$\frac{\partial \pi_i}{\partial q_i} = 1 - \gamma q_j - 2q_i; \qquad (37)$$

the best reply function and the resulting one-shot deviation profit obtain:

$$R_C(q) = \max\left\{\frac{1-\gamma q}{2}; 0\right\}; \qquad \pi_C^D(q) = (R_C(q))^2. \qquad (38)$$

Based upon (3), (36) and (38), deviation profits are calculated as

$$\pi_C^D(q^M) = \left(\frac{2+\gamma}{4(1+\gamma)}\right)^2; \qquad \pi_C^D(q^P) = \begin{cases} \left(\frac{1-\gamma q^P}{2}\right)^2 & \text{if } \frac{1-\gamma q^P}{2} > 0; \\ 0 & \text{if } \frac{1-\gamma q^P}{2} \le 0. \end{cases} \qquad (39)$$

On the other hand, the punishment profit is

$$\pi_C(q^P) = \left(1 - (1+\gamma)q^P\right) q^P \qquad (40)$$

which is not subject to a positivity constraint.

Using (3), (39) and (40), the system of simultaneous equations (18) and (19) is solved by:

$$q^P = \frac{2+3\gamma}{2(1+\gamma)(2+\gamma)}; \quad \delta_C^* = \frac{(2+\gamma)^2}{16(1+\gamma)}. \qquad (20)$$

This quantity, however, stays positive only over the range $\gamma \in \left[-\frac{2}{3}, 1\right]$. Therefore, over the range $\gamma \in \left(-1, -\frac{2}{3}\right)$, the optimal punishment rule proves itself technologically infeasible in sustaining collusion at q^M.

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